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Technical Memorandum

No. K-47/64

ANALYTIC SOLUTION OF A

SPECIAL GUN PROBLEM

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U. S. NAVAL WEAPONS LABORATORY

TECHNICAL MEMORANDUM

July 1964

No. K-47/64

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Computation and Analysis Laboratory

Approved by:

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While the contents of this memorandum are considered to be correct,
they are subject to modification upon further study.

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ABSTRACT

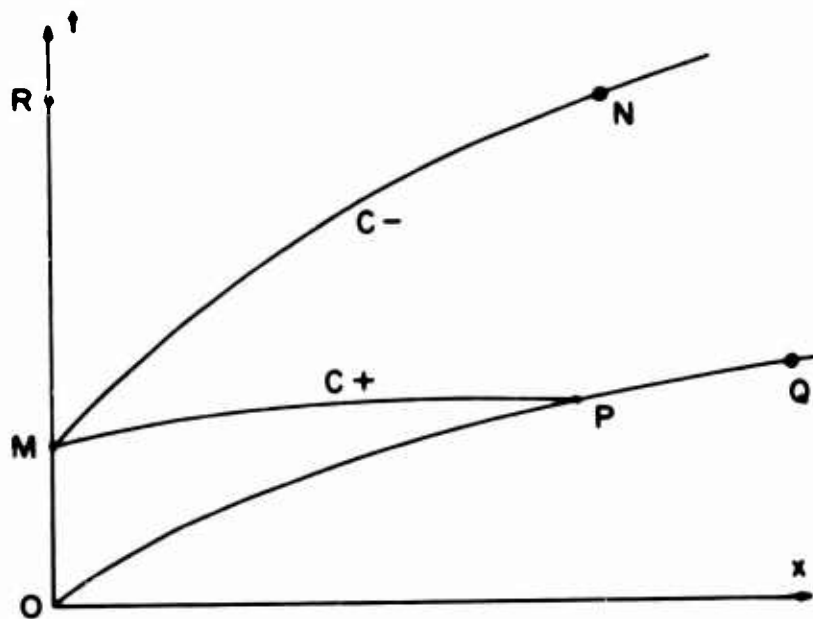
The motion of a projectile in a gun barrel is prescribed so that the motion of the gas in the part of the barrel adjacent to the projectile is given by a simple analytic function. Analytic methods involving the hodograph transformation are employed to determine the motion of the gas in the remaining portion of the barrel. The procedures are carried out explicitly in the case $\gamma = \frac{5}{3}$.

FOREWORD

The investigations of this report were performed in the Applied Mathematics Section of the Mathematics Research Group as a project supported under WEPTASK No. WR-4-0046. The investigations were undertaken in connection with a problem proposed by Dr. Arnold E. Seigel of the U. S. Naval Ordnance Laboratory. The date of completion was 1 July 1964.

I. INTRODUCTION

In the problem to be considered, a projectile P moves through a gun barrel with a prescribed motion, describing the path OPQ in the position vs. time plane. The motion of the gas is prescribed behind the projectile and at the base of the barrel such that the gas equations



have a simple analytic solution, as long as the gas velocity remains below the local sound speed at the base of the barrel (segment OM in the diagram). At point M , the gas velocity equals the local sound speed; it is prescribed that thereafter they remain equal (but not constant) at the base of the barrel (segment MR). The initial analytic solution is regarded as holding throughout the region $QOMN$ bounded by the projectile path and by the C^- -characteristic MN through M .

The problem then is to determine the motion of the gas in the remaining region NMR . An attempt is made in the following to obtain an analytic solution in this region; the details are carried out in the particular case $\gamma = 5/3$. It remains to be seen whether this solution is of unrestricted or of limited validity in this region, due to the occurrence of shocks or other discontinuities.

II. DESCRIPTION OF PROBLEM

The motion of the projectile, in normalized form, is

$$(1) \quad x = \frac{1}{2} t^2, \quad ,$$

$$(2) \quad u = t, \quad ,$$

for position and velocity respectively. (The original unnormalized problem, together with conversion factors, is stated in the appendix.) A polytropic gas lies behind the projectile; along the projectile path its pressure and sound speed remain at their initial values:

$$(3) \quad p = p_0 = 1, \quad ,$$

$$(4) \quad a = a_0 = \frac{t^{-1}}{2}, \quad ,$$

in normalized form.

At the base of the barrel, $x = 0$, the gas velocity is again

$$(5) \quad u = t, \quad ,$$

as long as this remains less than the local sound speed.

At time $t = t_M$ (point M), the velocity equals the sound speed; specifically we then have

$$(6) \quad u = a = t = t_M = \frac{t^{-1}}{2} \sqrt{\frac{2}{3-\gamma}}.$$

For $t > t_M$, the condition

$$(7) \quad u = a$$

is specified at $x = 0$ (along MR). (Note that this problem is meaningful for a polytropic gas for $1 < \gamma < 3$ only; this assumption is made throughout.)

The continuity and Euler gas equations are respectively

$$(8) \quad \sigma_t + u \sigma_x + \frac{t^{-1}}{2} \sigma u_x = 0$$

and

$$(9) \quad u_t + u u_x + \frac{t^{-1}}{2} \sigma \sigma_x = 0.$$

Here, for a polytropic gas,

$$(10) \quad \sigma = \frac{2}{\gamma-1} a$$

is the Riemann quantity, such that

$$(11) \quad u + \sigma = 2\beta$$

and

$$(12) \quad u - \sigma = -2\eta$$

are invariant along C^+ and C^- -characteristics respectively.

With our assumptions, we may take

$$(13) \quad u = t$$

throughout the region QOMN. The gas equations then reduce to

$$(14) \quad \sigma_t + t \sigma_x = 0$$

and

$$(15) \quad 1 + \frac{t-1}{2} \sigma \sigma_x = 0;$$

these have the general solutions

$$(16) \quad \sigma = f\left(x - \frac{t^2}{2}\right)$$

and

$$(17) \quad \sigma^2 + \frac{4}{\gamma-1} x = g(t)$$

($f(z)$ and $g(z)$ arbitrary functions) respectively. The resulting solution in QOMN is

$$(18) \quad \sigma(x, t) = \sqrt{1 - \frac{4}{\gamma-1} \left(x - \frac{t^2}{2}\right)}.$$

(At the projectile, $x = \frac{t^2}{2}$, we have $\sigma = \frac{2}{\gamma-1} a = 1$. Hence by (17),

$$g(t) = 1 + \frac{4}{\gamma-1} \left(\frac{t^2}{2}\right). \quad \text{Then by (17) again,}$$

$$\sigma = \sqrt{1 + \frac{4}{\gamma-1} \left(\frac{t^2}{2}\right) - \frac{4}{\gamma-1} x}, \quad \text{which agrees with (16).)$$

The C^- -characteristic MN through M must satisfy

$$(19) \quad u - \sigma = u_M - \sigma_M = \left(1 - \frac{2}{\gamma-1}\right) t_M;$$

this yields

$$(20) \quad x = \frac{1}{2} \left[\sqrt{\frac{3-\gamma}{2}} t - \frac{\gamma-1}{2} \right]^2 = \frac{3-\gamma}{4} \left[t - \frac{\gamma-1}{2} \sqrt{\frac{2}{3-\gamma}} \right]^2$$

as the equation for MN.

The problem now becomes that of determining $u(x,t)$ and $\sigma(x,t)$ in the region NMR, such that they satisfy the gas equations (8) and (9) within the region, and such that the initial conditions

$$(21) \quad u = a = \frac{\gamma-1}{2} \sigma$$

hold along $x = 0$ and (the consistent conditions)

$$(22) \quad u = t, \quad \sigma = t + \sqrt{\frac{3-\gamma}{2}}$$

$$\text{hold along } x = \frac{3-\gamma}{4} \left[t - \frac{\gamma-1}{2} \sqrt{\frac{2}{3-\gamma}} \right]^2.$$

III. METHOD OF SOLUTION

The solution of the problem described above will be attempted by means of the hodograph transformation. From (11) and (12), we obtain

$$(23) \quad u = \xi - \eta,$$

$$(24) \quad \sigma = \xi + \eta,$$

in terms of the characteristic coordinates (ξ, η) , where ξ is a constant along any C^+ -characteristic and η is a constant along any C^- -characteristic.

In these coordinates, the gas equations become equivalent to the equations

$$(25) \quad x_\xi = \left(\frac{3-\gamma}{2} \xi - \frac{\gamma+1}{2} \eta \right) t_\xi$$

and

$$(26) \quad x_\eta = \left(\frac{\gamma+1}{2} \xi - \frac{3-\gamma}{2} \eta \right) x_\xi.$$

Let

$$(27) \quad \lambda = \frac{1}{2} \left(\frac{\gamma+1}{\gamma-1} \right);$$

upon differentiation of the above equations with respect to η and ξ respectively, and elimination of $x_{\xi\eta}$, a linear equation for $x(\xi, \eta)$ alone results:

$$(28) \quad x_{\xi\eta} + \frac{\lambda}{\xi+\eta} (x_\xi + x_\eta) = 0.$$

In the (ξ, η) -plane, the C^- and C^+ -characteristics are horizontal and vertical lines respectively, while the region NMR now is

bounded by the straight lines $\frac{1}{2} \sqrt{\frac{3-\gamma}{2}}$
 MN: $\eta = \frac{1}{2} \sqrt{\frac{3-\gamma}{2}}$ and MR:

$$\eta = \frac{3-\gamma}{\gamma+1} \xi. \quad \text{The}$$

initial conditions to be satisfied by the solutions of (25) and (26) are

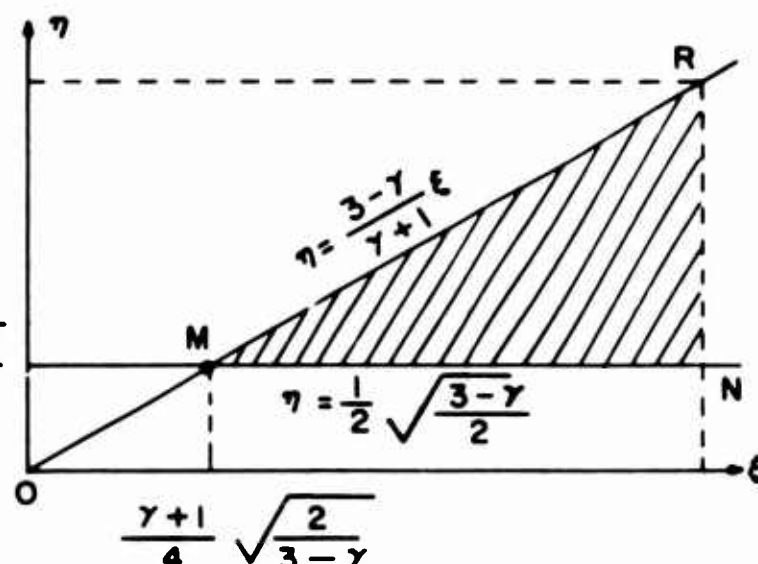
$$(29) \quad x = 0$$

along line MR, and (the consistent conditions)

$$(30) \quad x = \xi - \frac{1}{2} \sqrt{\frac{3-\gamma}{2}}, \quad x = \frac{3-\gamma}{4} \left[\xi - \frac{\gamma+1}{4} \sqrt{\frac{2}{3-\gamma}} \right]^2$$

along the C^- -characteristic MN.

Equation (28) for $x(\xi, \eta)$ is one for which the Riemann function is known (Reference [2] pp.449-461), thus enabling the solutions of normal initial-value problems to be written in explicit form. The Riemann function



may be written in either of the forms

$$(31) \quad R_\lambda(x, y; \xi, \eta) = \left(\frac{x+y}{\xi+\eta}\right)^\lambda F\left[\begin{matrix} 1-\lambda, \lambda \\ 1 \end{matrix}; -\frac{(x-\xi)(y-\eta)}{(x+\eta)(\xi+\eta)}\right] \\ = \frac{(x+y)^{2\lambda}}{(x+\eta)^\lambda(y+\xi)^\lambda} F\left[\begin{matrix} \lambda, \lambda \\ 1 \end{matrix}; \frac{(x-\xi)(y-\eta)}{(x+\eta)(y+\xi)}\right],$$

where $F(a, b; c; z)$ is the Gaussian hypergeometric function. (Note that

$$(32) \quad F(a, b; c; z) = (1-z)^{-b} F(c-a, b; c; -\frac{z}{1-z}),$$

from which the equivalence of the two forms follows.) When λ is an integer, and thus in particular for the important cases $\lambda = \frac{5}{3}$ ($\lambda = 2$) and $\lambda = \frac{7}{3} = 1.4$ ($\lambda = 3$), the Riemann function is rational in its four variables. Specifically, for $\lambda = \frac{5}{3}$ we find that

$$(33) \quad R_2(x, y; \xi, \eta) = \frac{x+y}{(\xi+\eta)^3} [2(xy+\xi\eta) + (x-y)(\xi-\eta)].$$

When λ is an integer, we may also write the general solution of (28) explicitly in the form (Reference [1], p. 90)

$$(34) \quad \mathcal{L}(\xi, \eta) = K + \frac{\partial^{1-\lambda}}{\partial \xi^{1-\lambda}} \left[\frac{f(\xi)}{(\xi+\eta)^\lambda} \right] + \frac{\partial^{1-\lambda}}{\partial \eta^{1-\lambda}} \left[\frac{g(\eta)}{(\xi+\eta)^\lambda} \right],$$

where the constant K and the functions $f(z)$ and $g(z)$ are arbitrary. This furnishes an alternate method of solution for these cases.

Our initial-value problem is abnormal in that it is partly characteristic and partly non-characteristic, and in that only $\mathcal{L}(\xi, \eta)$ is known along the non-characteristic part of the boundary. This complicates the method of solution if the general Riemann method is to be used to solve (28) directly.

IV. THE SOLUTION FOR THE CASE $\gamma = 5/3$

The solution has been carried out explicitly for the case $\gamma = \frac{5}{3}$ ($\lambda = 2$), the simplest non-trivial case. The methods used here could be extended immediately to the other cases with integral λ . The Riemann function method in principle is extendable to the case with λ arbitrary, but some other method for solving the integro-differential equation (75) for $T(\eta)$, i.e. $t(\xi, \eta)$ along MR, would appear to be required then.

The solution obtained here is in the hodograph plane and is valid throughout all of NMR there. However the range of its validity in the original plane remains to be determined; this depends on the mapping into the original plane remaining one-to-one, i.e. on the characteristics of the same family not intersecting within NMR in the (x, t) -plane. This can readily be investigated numerically by using the solutions derived here; a short computer program for this purpose is now being written. (The line MR in the original plane is known to be singular, however.)

The solution was first obtained by using the general solution (34). As this method is limited to integral λ , an attempt was then made to use the Riemann function method. With the latter method, the problem separated into two steps; the first step led to an integro-differential equation essentially for the missing initial value $t(\xi, \eta)$ along MR. The second step is then the ordinary non-characteristic Riemann solution, with the above initial value; this is straightforward and is not carried out here in detail (see Reference [2], pp. 449-453).

A. General Solution Method

For $\eta = \frac{\xi}{3}$, equations (25), (26) and (28) become

$$(35) \quad x_{\xi} = \frac{2}{3}(\xi - 2\eta) x_{\eta},$$

$$(36) \quad x_{\eta} = \frac{2}{3}(2\xi - \eta) x_{\xi},$$

and

$$(37) \quad x_{\xi\eta} + \frac{2}{\xi+\eta}(x_{\xi} + x_{\eta}) = 0.$$

The initial conditions become

$$(38) \quad x = 0$$

along line MR: $\xi = 2\eta$,

and

$$(39) \quad x = \xi - \frac{\sqrt{6}}{6}, \quad x = \frac{1}{3}\left(\xi - \frac{\sqrt{6}}{3}\right)^2$$

along NM: $\eta = \frac{\sqrt{6}}{6}$. The point M has coordinates

$$(40) \quad \xi = \frac{\sqrt{6}}{3}, \quad \eta = \frac{\sqrt{6}}{6};$$

here the initial values are

$$(41) \quad x = 0, \quad x = \frac{\sqrt{6}}{6}.$$

From (34), the general solution for $t(\xi, \eta)$ is

$$(42) \quad t(\xi, \eta) = \frac{1}{(\xi+\eta)^2} \left\{ (\xi+\eta)[f'(\xi) + g'(\eta)] - 2[f(\xi) + g(\eta)] \right\},$$

the constant K of (34) turns out to be superfluous.

Using (35) and (36), we find for $x(\xi, \eta)$ the expressions

$$(43) \quad x = \int x_{\xi} d\xi = \frac{2}{3} \int (\xi - 2\eta) x_{\eta} d\xi \\ = C' + \frac{2}{3} \left[(\xi - 2\eta) x(\xi, \eta) + \frac{2}{2\eta} \left[\frac{g(\eta)}{\xi + \eta} \right] - \frac{f(\xi)}{(\xi + \eta)^2} \right],$$

and

$$(44) \quad x = \int x_{\eta} d\eta = \frac{2}{3} \int (2\xi - \eta) x_{\eta} d\eta$$

$$= C' + \frac{2}{3} \left[(2\xi - \eta) x(\xi, \eta) - \frac{\partial}{\partial \xi} \left[\frac{f(\xi)}{\xi + \eta} \right] + \frac{g(\eta)}{(\xi + \eta)^2} \right].$$

(Here C' and C'' are arbitrary functions of η and ξ respectively.)

The two expressions must be the same, hence $C' = C'' = C$, a constant.

Working out the details, we obtain finally

$$(45) \quad x(\xi, \eta) = \frac{1}{(\xi + \eta)^3} \left\{ C(\xi + \eta)^3 + \frac{2}{3}(\xi + \eta) \left[(\xi - 2\eta) f'(\xi) \right. \right.$$

$$\left. \left. + (2\xi - \eta) g'(\eta) \right] - 2(\xi - \eta) [f(\xi) + g(\eta)] \right\}.$$

(The constant C will later turn out to be superfluous.)

On MR: $\xi = 2\eta$, we have from (38) the result

$$(46) \quad 0 = x(2\eta, \eta) = \frac{1}{(3\eta)^3} \left\{ C(3\eta)^3 + \frac{2}{3}(3\eta)^2 g'(\eta) - 2\eta [f(2\eta) + g(\eta)] \right\},$$

and so the differential equation for $g(\eta)$:

$$(47) \quad g'(\eta) - \frac{1}{3\eta} g(\eta) = -\frac{2}{3} C\eta + \frac{1}{3\eta} f(2\eta).$$

The general solution of this equation is

$$(48) \quad g(\eta) = K\eta^{1/3} - \frac{27}{10} C\eta^2 + \frac{1}{3} \eta^{1/3} \int \eta^{-4/3} f(2\eta) d\eta.$$

On NM: $\eta = \frac{\sqrt{6}}{6}$, we have

$$(49) \quad x\left(\xi, \frac{\sqrt{6}}{6}\right) = \xi - \frac{\sqrt{6}}{6} = \frac{1}{(\xi + \frac{\sqrt{6}}{6})^3} \left\{ \left(\xi + \frac{\sqrt{6}}{6}\right) \left[f'(\xi) + g'\left(\frac{\sqrt{6}}{6}\right) \right] \right.$$

$$\left. - 2 \left[f(\xi) + g\left(\frac{\sqrt{6}}{6}\right) \right] \right\},$$

and

$$(50) \quad x\left(\mathfrak{F}, \frac{\sqrt{6}}{6}\right) = \frac{1}{3}\left(\mathfrak{F} - \frac{\sqrt{6}}{3}\right)^2 = \frac{1}{\left(\mathfrak{F} + \frac{\sqrt{6}}{6}\right)^3} \left\{ C\left(\mathfrak{F} + \frac{\sqrt{6}}{6}\right)^3 + \frac{2}{3}\left(\mathfrak{F} + \frac{\sqrt{6}}{6}\right) \left[\left(\mathfrak{F} - \frac{\sqrt{6}}{3}\right) f'(\mathfrak{F}) + \left(2\mathfrak{F} - \frac{\sqrt{6}}{6}\right) g'\left(\frac{\sqrt{6}}{6}\right) - 2\left(\mathfrak{F} - \frac{\sqrt{6}}{6}\right) \left[f(\mathfrak{F}) + g\left(\frac{\sqrt{6}}{6}\right) \right] \right] \right\}.$$

This pair of linear algebraic equations for $f(\mathfrak{F})$ and $f'(\mathfrak{F})$ may be solved to give

$$(51) \quad f(\mathfrak{F}) = \frac{1}{2}\left(\mathfrak{F} + \frac{\sqrt{6}}{6}\right)^2 \left[3C + \mathfrak{F}^2 - \frac{\sqrt{6}}{3}\mathfrak{F} \right] + \left(\mathfrak{F} + \frac{\sqrt{6}}{6}\right) g'\left(\frac{\sqrt{6}}{6}\right) - g\left(\frac{\sqrt{6}}{6}\right),$$

and

$$(52) \quad f'(\mathfrak{F}) = \left(\mathfrak{F} + \frac{\sqrt{6}}{6}\right) \left[3C + 2\mathfrak{F}^2 - \frac{\sqrt{6}}{3}\mathfrak{F} - \frac{1}{6} \right] + g'\left(\frac{\sqrt{6}}{6}\right);$$

this solution may be verified by differentiating (51).

The constants $g\left(\frac{\sqrt{6}}{6}\right)$ and $g'\left(\frac{\sqrt{6}}{6}\right)$ prove to be superfluous, and we may set

$$(53) \quad g\left(\frac{\sqrt{6}}{6}\right) = g'\left(\frac{\sqrt{6}}{6}\right) = 0.$$

Then, substituting (51) with $\mathfrak{F} = 2\eta$ into (48) gives

$$(54) \quad g(\eta) = K\eta^{1/3} - \frac{1}{2}C(3\eta^2 - \sqrt{6}\eta + \frac{1}{2}) + \frac{8}{11}\eta^4 - \frac{1}{5}\eta^2 - \frac{\sqrt{6}}{36}\eta,$$

and differentiating gives

$$(55) \quad g'(\eta) = \frac{K}{3}\eta^{-2/3} - \frac{1}{2}C(6\eta - \sqrt{6}) + \frac{32}{11}\eta^3 - \frac{2}{5}\eta - \frac{\sqrt{6}}{36}.$$

Setting $\eta = \frac{\sqrt{6}}{6}$ and using (53) in either of the above yields

$$(56) \quad K = \frac{9}{220}\sqrt{6};$$

C proves to be superfluous and may be set equal to zero.

Thus we obtain:

$$(57) f(\xi) = \frac{1}{2}\xi(\xi - \frac{\sqrt{6}}{3})/(\xi + \frac{\sqrt{6}}{6})^2 = \frac{1}{2}\xi^2 - \frac{1}{4}\xi^2 - \frac{\sqrt{6}}{36}\xi,$$

$$(58) f'(\xi) = 2\xi^2 - \frac{1}{2}\xi - \frac{\sqrt{6}}{36},$$

$$(59) g(\eta) = \frac{9}{220}\sqrt{6}\eta^{1/3} + \frac{8}{11}\eta^4 - \frac{1}{5}\eta^2 - \frac{\sqrt{6}}{36}\eta,$$

and

$$(60) g'(\eta) = \frac{3}{220}\sqrt{6}\eta^{-2/3} + \frac{32}{11}\eta^3 - \frac{2}{5}\eta - \frac{\sqrt{6}}{36}.$$

From (42) and (45) we now obtain the solutions within NMR:

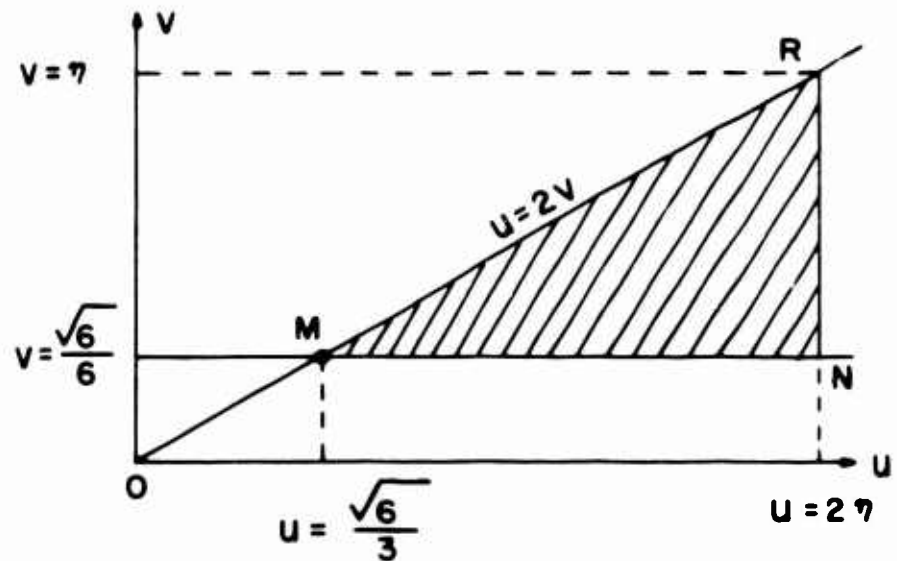
$$(61) x(\xi, \eta) = \frac{1}{(\xi + \eta)^2} \left[\frac{3}{220}\sqrt{6}(\xi - 5\eta)\eta^{-2/3} + \xi^4 + \frac{16}{11}\eta^4 \right. \\ \left. + 2\xi\eta(\xi^2 + \frac{16}{11}\eta^2) - \frac{9}{10}\xi\eta \right],$$

and

$$(62) x(\xi, \eta) = \frac{(\xi - 2\eta)^2}{(\xi + \eta)^3} \left[\frac{\sqrt{6}}{55}\eta^{-2/3} + \frac{1}{3}\xi^3 + \xi^2\eta - \frac{4}{33}\eta^3 + \frac{1}{6}\xi - \frac{1}{30}\eta \right].$$

B. Riemann Function Method

Riemann's solution for the noncharacteristic initial-value problem gives the solution $t(\xi, \eta)$ at a point N in terms of the values of t and its derivatives along MR. Our problem is an inverse one, in that the value of $t(\xi, \eta)$ at



N is known while that along MR is partly unknown. The result is that the solution formula will constitute an integro-differential equation for the function t along MR.

Once t has thus been determined, the Riemann solution may again be employed, this time directly, to obtain the solution for arbitrary points inside NMR.

The symmetric form of the Riemann solution for our problem is

$$(63) \quad t(2\eta, \frac{\sqrt{6}}{6}) = \frac{1}{2} \left[R_2(\frac{\sqrt{6}}{3}, \frac{\sqrt{6}}{6}; 2\eta, \frac{\sqrt{6}}{6}) t(\frac{\sqrt{6}}{3}, \frac{\sqrt{6}}{6}) \right. \\ \left. + R_2(2\eta, \eta; 2\eta, \frac{\sqrt{6}}{6}) t(2\eta, \eta) \right] + I(\eta),$$

where $I(\eta)$ is the line integral along MR:

$$(64) \quad I(\eta) = \int_{MR} \left\{ \left[\frac{R_2}{2} t_u(u, v) + \left(\frac{2R_2}{u+v} - \frac{R_u}{2} \right) t(u, v) \right] du \right. \\ \left. - \left[\frac{R_2}{2} t_v(u, v) + \left(\frac{2R_2}{u+v} - \frac{R_v}{2} \right) t(u, v) \right] dv \right\},$$

where R_2 is the Riemann function

$$(65) \quad R_2 = R_2(u, v; 2\eta, \frac{\sqrt{6}}{6})$$

and $R_u = (R_2)_u$, $R_v = (R_2)_v$ for short.

(See Reference [2], page 453, equation (4), where however the term $a R_v$ should be just $a R$.)

The line integral may be evaluated by setting

$$(66) \quad u = 2\tau, \quad v = \tau, \quad \frac{\sqrt{6}}{6} \leq \tau \leq \eta.$$

This gives

$$(67) \quad I(\eta) = \int_{\frac{\sqrt{6}}{6}}^{\eta} \left\{ R_2(2\tau, \tau; 2\eta, \frac{\sqrt{6}}{6}) \left[t_u(2\tau, \tau) - \frac{1}{2} t_v(2\tau, \tau) \right] \right. \\ \left. + \left[\frac{2}{3\tau} R_2(2\tau, \tau; 2\eta, \frac{\sqrt{6}}{6}) - R_u(2\tau, \tau; 2\eta, \frac{\sqrt{6}}{6}) + \frac{1}{2} R_v(2\tau, \tau; 2\eta, \frac{\sqrt{6}}{6}) \right] t(2\tau, \tau) \right\} d\tau.$$

The function $t(2\tau, \tau)$ along MR is essentially a function of one variable; hence set

$$(68) \quad T(\tau) = t(2\tau, \tau).$$

Along MR we are given that (38)

$$(69) \quad x(2\tau, \tau) \equiv 0;$$

hence also

$$(70) \quad \frac{d}{d\tau} x(2\tau, \tau) = 2x_u(2\tau, \tau) + x_v(2\tau, \tau) \equiv 0.$$

From the gas equations, (35) gives

$$(71) \quad x_u(2\tau, \tau) \equiv 0,$$

and (36) and (70) together then give

$$(72) \quad x_v(2\tau, \tau) = 2\tau t_v(2\tau, \tau) \equiv 0.$$

Thus, since τ is not identically zero,

$$(73) \quad t_v(2\tau, \tau) \equiv 0.$$

Therefore,

$$(74) \quad T'(\tau) = \frac{d}{d\tau} t(2\tau, \tau) = 2t_u(2\tau, \tau) + t_v(2\tau, \tau) = 2t_u(2\tau, \tau).$$

Thus the knowledge of $x(2\tau, \tau)$ enables us to replace the derivatives of t along MR by the single derivative $T'(\tau)$; (63) then becomes finally

$$(75) \quad t(2\eta, \frac{\sqrt{E}}{6}) = \frac{1}{2} \left[R_2(\frac{\sqrt{E}}{3}, \frac{\sqrt{E}}{6}; 2\eta, \frac{\sqrt{E}}{6}) T(\frac{\sqrt{E}}{6}) + R_2(2\eta, \eta; 2\eta, \frac{\sqrt{E}}{6}) T(\eta) \right] \\ + \int_{\frac{\sqrt{E}}{6}}^{\eta} \left\{ \frac{1}{2} R_2(2\tau, \tau; 2\eta, \frac{\sqrt{E}}{6}) T'(\tau) + \left[\frac{2}{3\tau} R_2(2\tau, \tau; 2\eta, \frac{\sqrt{E}}{6}) \right. \right. \\ \left. \left. - R_u(2\tau, \tau; 2\eta, \frac{\sqrt{E}}{6}) + \frac{1}{2} R_v(2\tau, \tau; 2\eta, \frac{\sqrt{E}}{6}) \right] T(\tau) \right\} d\tau.$$

This is the desired integro-differential equation for $T(\eta)$.

The Riemann function and its derivatives for $\lambda = 2$ are given explicitly by (33):

$$(76) R_2(2\tau, \tau; 2\eta, \frac{\sqrt{6}}{6}) = \frac{3\tau}{(2\eta + \frac{\sqrt{6}}{6})^3} \left[4\tau^2 + (2\eta - \frac{\sqrt{6}}{6})\tau + \frac{2}{3}\sqrt{6}\eta \right],$$

$$(77) R_u(2\tau, \tau; 2\eta, \frac{\sqrt{6}}{6}) = \frac{2}{(2\eta + \frac{\sqrt{6}}{6})^3} \left[5\tau^2 + 2(2\eta - \frac{\sqrt{6}}{6})\tau + \frac{1}{3}\sqrt{6}\eta \right],$$

and

$$(78) R_v(2\tau, \tau; 2\eta, \frac{\sqrt{6}}{6}) = \frac{2}{(2\eta + \frac{\sqrt{6}}{6})^3} \left[8\tau^2 - (2\eta - \frac{\sqrt{6}}{6})\tau + \frac{1}{3}\sqrt{6}\eta \right].$$

Also, we have from (39) the values

$$(79) \chi(2\eta, \frac{\sqrt{6}}{6}) = 2\eta - \frac{\sqrt{6}}{6}$$

and

$$(80) T(\frac{\sqrt{6}}{6}) = \chi(\frac{\sqrt{6}}{3}, \frac{\sqrt{6}}{6}) = \frac{\sqrt{6}}{6}.$$

Equation (75) can then be written in the form

$$(81) \int_{\frac{\sqrt{6}}{6}}^{\eta} \left\{ \left[6\tau^3 + (3\eta - \frac{\sqrt{6}}{6})\tau^2 + \sqrt{6}\eta\tau \right] T'(\tau) \right. \\ \left. + \left[6\tau^2 - (6\eta - \frac{\sqrt{6}}{2})\tau + \sqrt{6}\eta \right] T(\tau) \right\} d\tau \\ + (9\eta^3 + \frac{3}{4}\sqrt{6}\eta^2) T(\eta) = (2\eta + \frac{\sqrt{6}}{6})^3 (2\eta - \frac{\sqrt{6}}{6}) - \frac{1}{4}(\sqrt{6}\eta + \frac{1}{2}).$$

The term in $T'(\tau)$ can be eliminated under the integral sign by an integration by parts; this yields

$$(82) \int_{\frac{\sqrt{6}}{6}}^{\eta} \left[6\tau^3 + (3\eta - \frac{\sqrt{6}}{6})\tau^2 + \sqrt{6}\eta\tau \right] T'(\tau) d\tau = (9\eta^3 + \frac{3}{4}\sqrt{6}\eta^2) T(\eta) \\ - \frac{1}{4}(\sqrt{6}\eta + \frac{1}{2}) - \int_{\frac{\sqrt{6}}{6}}^{\eta} \left[18\tau^2 + (6\eta - \frac{\sqrt{6}}{2})\tau + \sqrt{6}\eta \right] T(\tau) d\tau.$$

Substituting into (81) gives the integral equation

$$(83) \quad \int_{\frac{\sqrt{6}}{6}}^{\eta} [12\tau^2 + (12\eta - \sqrt{6})\tau] T(\tau) d\tau - (18\eta^3 + \frac{8}{3}\sqrt{6}\eta) T(\eta) \\ = - (2\eta + \frac{\sqrt{6}}{6})^3 (2\eta - \frac{\sqrt{6}}{6}).$$

As the coefficient of $T(\tau)$ under the integral sign is a polynomial in η , (83) can be reduced to a differential equation for $T(\eta)$ by successive differentiations with respect to η . Here two differentiations are required, which yield the Euler equation

$$(84) \quad T''(\eta) + \frac{14}{3\eta} T'(\eta) + \frac{8}{3\eta^2} T(\eta) = \frac{32}{3\eta}.$$

As initial conditions, we have (80), and from (83) once differentiated, also

$$(85) \quad T'(\frac{\sqrt{6}}{6}) = 2.$$

The solution of (84) satisfying (80) and (85) is

$$(86) \quad T(\eta) = \frac{1}{11}(2\eta, \eta) = \frac{16}{11}\eta - \frac{1}{15}\eta^{-1} - \frac{\sqrt{6}}{660}\eta^{-5/3}.$$

This is the required value of t along MR; it can be readily verified that it is identical with (61) with 2η substituted for \mathcal{F} .

The Riemann solution, with (86), (73) and (74), may now be used in the usual manner to obtain $t(\mathcal{F}, \eta)$ at arbitrary points (\mathcal{F}, η) within NMR.

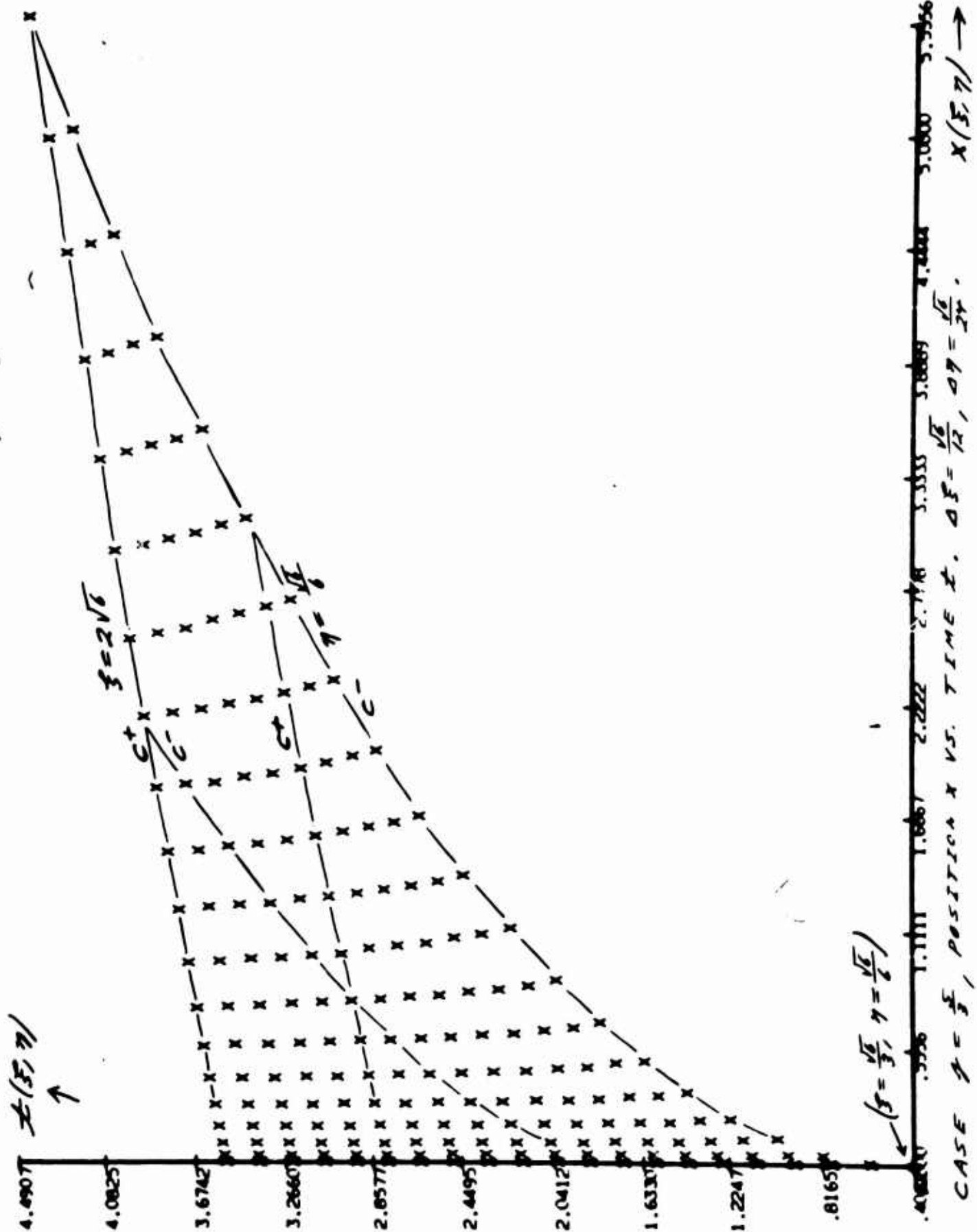
In the general case, where the Riemann function does not reduce to a rational function, the principal requirement for a solution by this method would appear to be a practical method for solving the integral equation corresponding to (83).

V. SOME NUMERICAL RESULTS

A short computer program has been written to evaluate and plot the formulas for x and t obtained in the preceding section. A plot where the data has been run out to $\xi = 2\sqrt{6}$ and $\eta = \sqrt{6}$ follows, with steps of $\Delta\xi = \frac{\sqrt{6}}{12}$ and $\Delta\eta = \frac{\sqrt{6}}{24}$. No singularities aside from the singular line $x = 0$ appear in the plot; the C^- -characteristics are all tangent to the line at $x = 0$. (In reading the plot, it should be noticed that an additional vertical line of points falls almost on top of the line $x = 0$, so that the parabola-shaped C^- -characteristics actually end one point below on $x = 0$ from their apparent end points.)

A plot with the data run out to $\xi = 25\sqrt{6}$, $\eta = \frac{25}{2}\sqrt{6}$ is very similar in appearance, with no singularities aside from $x = 0$ apparent. However, because of the non-linearity of the problem, it cannot be assumed from this alone that singularities inside the region of interest will not eventually appear. Additional runs with data run out to increasingly greater distances are planned.

$T(XS1, ETA) \text{ VS. } X(XS1, ETA) \quad \eta = \frac{5}{3}$



REFERENCES

- [1] R. Courant and K. O. Friedrichs, Supersonic Flow and Shock Waves. Interscience Publishers, 1948, pp. 79-91, 191-197.
- [2] R. Courant and D. Hilbert, Methods of Mathematical Physics, Volume II, "Partial Differential Equations". Interscience Publishers, 1962, pp. 429-432, 449-461.

APPENDIX A

APPENDIX A

The Unnormalized Problem

The problem as stated in the body of the report is in fully normalized form. The original form of the problem is stated below, following which is given a set of conversion ratios between the unnormalized and the normalized quantities.

The motion of the projectile is given as

$$(A1) \quad x = \frac{1}{2} \alpha t^2,$$

$$(A2) \quad u = \alpha t,$$

with

$$(A3) \quad \alpha = \frac{P_0 A}{M}.$$

Here P_0 is the initial gas pressure, A the cross-section area of the barrel, and M the mass of the projectile, in consistent units.

Let a_0 be the initial sound speed; at the base of the barrel the gas motion is given by

$$(A4) \quad u = \alpha t,$$

$$(A5) \quad a^2 = a_0^2 + \frac{\gamma-1}{2} \alpha^2 t^2,$$

as long as $u < a$, the local sound speed. At time

$$(A6) \quad t = t_M = \frac{a_0}{\alpha} \sqrt{\frac{2}{\gamma-1}}$$

the condition

$$(A7) \quad u = a$$

is attained; this condition holds at the base of the barrel for $t > t_M$.

The C⁻-characteristic NM bounding the prescribed solution is given by the equation

$$(A8) \quad x = \frac{a_0^2}{2\alpha} \left[\sqrt{\frac{\beta-1}{2}} \frac{\alpha t}{a_0} - 1 \right]^2;$$

along this characteristic the conditions

$$(A9) \quad u = \alpha t,$$

$$(A10) \quad \sigma = \frac{2}{\beta-1} a = \alpha t + \frac{\sqrt{2(\beta-1)}}{\beta-1} a_0$$

hold.

If f represents a given unnormalized quantity, let \bar{f} denote its normalized counterpart (the quantity used elsewhere in the report), and f^* the conversion factor connecting the two, so that

$$(A11) \quad f = (f^*) \bar{f}.$$

Define

$$(A12) \quad \sigma_0 = \frac{2}{\beta-1} a_0.$$

The conversion factors are now the following:

$$(A13) \quad t^* = \frac{\sigma_0}{\alpha},$$

$$(A14) \quad x^* = \frac{\sigma_0^2}{\alpha},$$

$$(A15) \quad u^* = a^* = \sigma^* = \sigma_0.$$